PG – 366

II Semester M.Sc. Degree Examination, June 2015
(CBCS)
MATHEMATICS
M 205 T : Functional Analysis

Time : 3 Hours Max. Marks : 70

***Instructions** : 1) Answer **any five full** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Define a normed linear space. Show that l_p^n is a normed linear space. **5**

b) Let M be a closed linear subspace of N. Prove that the quotient space $\frac{N}{M}$ is a normed linear space. **5**

c) Show that a linear transformation $T : N \to N'$ between normed linear spaces N and N' is continuous on N if and only if it is continuous at the origin. **4**

2. a) Let N be a non-zero normed linear space. Prove that N is a Banach space if and only if $S = \{x \in N \mid \|x\| = 1\}$ is compute. **5**

b) Let $B(N, N')$ be the vector space of all bounded linear transformations of a normed linear space N into a normed linear space N'. Show that $B(N, N')$ is a normed linear space and is compute when N' is compute. **6**

c) Show that each element in a normed linear space N gives rise to an element in N**. **3**

3. a) State and prove Hahn-Banach theorem for a real normed linear space. **7**

b) Let M be a closed linear subspace of a normed linear space N and let $x_0 \notin M$. If $d = d(x_0, M)$ then show that there is a functional h in N' such that $h(M) = 0$, $h(x_0) = 1$, $\|h\| = 1/d$. **4**

c) State the open mapping theorem. Use it to prove that a one-one continuous linear transformation of a Banach space B onto a Banach space B' is a homeomorphism. **3**

P.T.O.

4. a) If P is a projection on a Banach space $B = M \oplus N$, where M is the range space of $\underline{P}$ and N is the null space of $\underline{P}$, then prove that M and N are closed. **4**

   b) Prove that a non-empty subset X of a normed linear space N is bounded if and only if f(X) is bounded for each $f \in N^*$. **4**

   c) Let T be an operator on a normed linear space N then show that its conjugate T^* is an operator on N^* and the mapping $T \to T^*$ is an isometric isomorphism of B(N) into B(N*). **6**

5. a) In a Hilbert space H prove that parallelogram law and polarisation identity holds. **6**

   b) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm. **8**

6. a) If M is closed proper subspace of H then prove that there exists a non-zero vector Z_0 in H such that $Z_0 \perp M$. **5**

   b) If M and N are closed linear subspaces of H such that $M \perp N$ then show that $M + N$ is a closed linear subspace of H. **5**

   c) Show that every non-zero Hilbert space contains a complete orthonormal set. **4**

7. a) Define a adjoint of an operator T on H and prove the following : **4**

   i) $(T_1 + T_2)^* = T_1^* + T_2^*$

   ii) $\|T^*\| = \|T\|$

   iii) $\|T^*T\| = \|T\|^2 = \|TT^*\|$.

   b) If T is an operator on H then show that $T = 0$ if and only if $(T_x, x) = 0, \ \forall x \in H$. **4**

   c) Define a self adjoint operator. Prove that an operator T on a Hilbert space H is self adjoint if and only if $<T_x, x>$ is real $\forall x \in H$. **6**

8. a) Define a normal operator on Hilbert space H. Prove that T is normal if and only if $\|T_x\| = \|T^* x\|, \ \forall x \in H$. **4**

   b) Show that the following are equivalents : **4**

   i) $T^*T = I$

   ii) $(T_x, T_y) = (x, y), \ \forall \ x, y \in H$

   iii) $\|T_x\| = \|x\|, \ \forall x \in H$.

   c) If P is a projection on a Hilbert space H, with range M and null space N, then prove that $M \perp N$ if and only if P is self adjoint and in this care $N = M^{\perp}$. **6**